You Already Solve Differential Equations Using Symmetries

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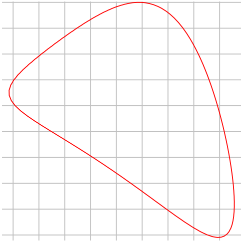
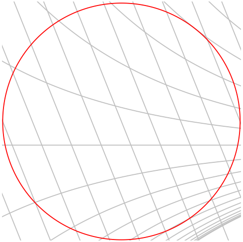




Outline

History of Symmetries and Differential Equations

- **1670s - 1690s** Newton and Leibniz develop, solve, and publish the first differential equations.
- **1700s - 1800s** Humans spend 200 years developing solution methods for differential equations.
- **1870s - 1880s** Sophus Lie proves that many solution methods are special cases of his symmetry method.
- **1910s** Emmy Noether proves profound connections between symmetries and conservation laws.
  (Einstein on Noether: "a creative mathematical genius.")
- **1980s - 1990s** Peter Olver, George Bluman, and others make symmetry methods accessible by publishing several excellent texts.
- **1990s - 2000s** Symmetry methods are incorporated into computer algebra systems, *e.g.*, Maple's `symgen` and Mathematica's `MathLie`.
- **2010s - 2020s** Numerical analysts begin to describe a systematic way of constructing symmetry preserving numerical methods for DE.

ODE

- ◦ Reduction of Order for Linear ODE
- ◦ Variation of Parameters for Linear ODE
- ◦ Superposition Principle for Linear ODE*
- ◦ Separation of Variables
- ◦ Integration of Exact ODE

PDE

- ◦ Separation of Variables for Linear PDE
- ◦ Superposition Principle for Linear PDE*
- ◦ Similarity Transformations
- ◦ D'Alembert's Formula
- ◦ Conformal Mapping
- ? The Method of Characteristics
- ? Separation of Variables for Nonlinear PDE

* Superposition symmetry enables linear transforms (Fourier, Laplace, Hankle).

Symmetry of an Object

An **object** could be a set, a graph, an equation, a function, a vector, or any other mathematical entity.

A **transformation** is an operation applied to the object which maps it to another object, or possibly the same one.

e.g., a vector $\vec{v}$ is transformed by a matrix M into another vector $\vec{w}$ by the operation of matrix multiplication, $\vec{w} = M\vec{v}$.

A **symmetry** of an object is a transformation which leaves the object unchanged. That is, if x is a given object, and $T(x)$ a particular transformation of that object, then T is a symmetry of x if and only if $T(x) = x$. When $T(x) = x$, we say that x is **invariant** under the transformation T or that T "leaves x invariant".

Example: If C is a circle centered at the origin, then the transformation which rotates points clockwise about the origin by a constant θ radians leaves C invariant.

Example: If R is the ratio x/y, then the coordinate scaling $(x, y) \to (sx, sy)$, leaves R invariant for all constants $s \neq 0$.

Example: If D is the differential equation $y'' = f(x, y')$ then the translation $(x, y) \to (x, y + c)$, leaves D invariant for any constant c. That is, the differential equation is invariant under translations in y.

Lie Point Symmetries of a Function

A **point transformation** is a continuous mapping $x \to \tilde{x}(x, \epsilon)$ with $\tilde{x}(x, 0) = x$.

If f is a function, and $x \to \tilde{x}(x, \epsilon)$ is a point transformation, then the function f is said to be **invariant** under the transformation if $f(\tilde{x}(x, \epsilon)) = f(x)$ for all ϵ sufficiently close to 0. We call this a **Lie point symmetry** of f.

Example: Define $f(x, y) = x^2 + y^2$ and let the transformation be given by

$$(\tilde{x}, \tilde{y}) = (x \cos \epsilon - y \sin \epsilon, x \sin \epsilon + y \cos \epsilon)$$

This is a Lie point symmetry of f because setting $\epsilon = 0$ gives (x, y) and

$$\begin{aligned} f(\tilde{x}, \tilde{y}) &= (x \cos \epsilon - y \sin \epsilon)^2 + (x \sin \epsilon + y \cos \epsilon)^2 \\ &= (x^2 + y^2)(\cos^2 \epsilon + \sin^2 \epsilon) = x^2 + y^2 = f(x, y) \end{aligned}$$

If $f(\tilde{x}) = 0$ whenever $f(x) = 0$, then the equation $f(x) = 0$ is said to be **invariant** under $\tilde{x}$, and $\tilde{x}$ is said to be a **Lie point symmetry** of the equation.

Remark: $\tilde{x}$ need not be a symmetry of the function f in order to be a symmetry of the equation $f = 0$.

Example: Consider the equation $f = 0$ where $f(x, y) = \frac{x-y}{y^2+x^2}$. Define a transformation.

$$\tilde{x} = (1 + \epsilon x)x, \quad \tilde{y} = (1 + \epsilon x)y$$

Thus

$$f(\tilde{x}, \tilde{y}) = \frac{\tilde{x} - \tilde{y}}{\tilde{x}^2 + \tilde{y}^2} = \frac{1 + \epsilon(x + y)}{(1 + \epsilon x)^2} \frac{x - y}{x^2 + y^2} = \frac{1 + \epsilon(x + y)}{(1 + \epsilon x)^2} f(x, y)$$

If $\epsilon x \neq -1$, the transformation is a Lie point symmetry of the equation $f = 0$ but not of the function f.

Suppose the transformation $(x, y) \to (\tilde{x}(x, y, \epsilon), \tilde{y}(x, y, \epsilon))$ is differentiable in ϵ. Linearize any smooth $f(\tilde{x}, \tilde{y})$ around $\epsilon = 0$.

$$f(\tilde{x}, \tilde{y}) = f(x, y) + \epsilon \left(\frac{\partial f}{\partial x}(x, y) \frac{\partial \tilde{x}}{\partial \epsilon}(x, y, 0) + \frac{\partial f}{\partial y}(x, y) \frac{\partial \tilde{y}}{\partial \epsilon}(x, y, 0) \right) + o(\epsilon)$$

A necessary condition for the transformation to be a symmetry is that the term in parenthesis vanishes.

Definition: The **infinitesimal generator** of the transformation $(\tilde{x}, \tilde{y})$ is

$$\chi = \frac{\partial \tilde{x}}{\partial \epsilon} \frac{\partial}{\partial x} + \frac{\partial \tilde{y}}{\partial \epsilon} \frac{\partial}{\partial y}$$

where derivatives are evaluated at $\epsilon = 0$.

Note that $\chi f = 0$ is a necessary condition for symmetry.
What's remarkable is that this condition is also sufficient.

Theorem (Lie, 1888)

If a point transformation, $\tilde{x}$, is differentiable in ϵ in a neighborhood of $\epsilon = 0$, then it is a Lie point symmetry of the differentiable function f in this neighborhood iff $\chi f = 0$. Moreover, the equation $f = 0$ is invariant iff $\chi f = 0$ whenever $f = 0$.

Example: The transformation $(\tilde{x}, \tilde{y}) = ((1+\epsilon)x, (1+2\epsilon)y)$ shears the plane. It's infinitesimal generator is

$$\chi = x\partial_x + 2y\partial_y$$

All differentiable functions invariant under this shearing satisfy

$$xf_x + 2yf_y = 0$$

and it is possible to show (method of characteristics) that all such functions have the form

$$f(x, y) = g(y/x^2)$$

The transformation $x \to \tilde{x}(x, y, \epsilon)$ and $y \to \tilde{y}(x, y, \epsilon)$ can be **prolonged** to the derivative dy/dx using the chain rule.

$$\frac{dy}{dx} \to \frac{d\tilde{y}}{d\tilde{x}} = \frac{\frac{\partial \tilde{y}}{\partial x} + \frac{\partial \tilde{y}}{\partial y}\frac{dy}{dx}}{\frac{\partial \tilde{x}}{\partial x} + \frac{\partial \tilde{x}}{\partial y}\frac{dy}{dx}}$$

If the differential equation $\frac{dy}{dx} = f(x, y)$ is invariant under the transformation then we say that the transformation is a Lie point symmetry of the differential equation.

Example: All first order linear homogeneous ODE, $y' = f(x)y$, are invariant under a scaling transformation $(x, y) \to (x, (1+\epsilon)y)$.

$$\tilde{x} = x, \quad \tilde{y} = (1+\epsilon)y, \quad \frac{d\tilde{y}}{d\tilde{x}} = \frac{0 + (1+\epsilon)\frac{dy}{dx}}{1 + 0\frac{dy}{dx}} = (1+\epsilon)\frac{dy}{dx}$$

so

$$y' - f(x)y \quad \longrightarrow \quad (1+\epsilon)(y' - f(x)y)$$

The prolongation of the Infinitesimal Generator

Suppose the transformation $(\tilde{x}, \tilde{y})$ has infinitesimal generator $\chi = \xi\partial_x + \eta\partial_y$, then the first prolongation of χ is found by computing the infinitesimal generator of the prolongation.

$$\frac{d\tilde{y}}{d\tilde{x}} = \frac{\frac{\partial\tilde{y}}{\partial x} + \frac{\partial\tilde{y}}{\partial y}\frac{dy}{dx}}{\frac{\partial\tilde{x}}{\partial x} + \frac{\partial\tilde{x}}{\partial y}\frac{dy}{dx}} = \frac{\epsilon\eta_x + (1+\epsilon\eta_y)y' + o(\epsilon)}{1 + \epsilon\xi_x + \epsilon\xi_y y' + o(\epsilon)}$$

$$= y' + \epsilon(\eta_x + \eta_y y' - y'(\xi_x + \xi_y y')) + o(\epsilon)$$

Therefor, the first prolongation $\chi^{(1)}$ is

$$\chi^{(1)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'}$$

where

$$\eta^{(1)} = \eta_x + \eta_y y' - y'(\xi_x + \xi_y y')$$

Remark: $\chi^{(1)}$ is the infinitesimal generator of the prolongation $(\tilde{x}, \tilde{y}, \frac{d\tilde{y}}{d\tilde{x}})$. As such, a version of Lie's theorem now applies to functions of the form $f(x, y, y')$.

Theorem (Lie 1888)

The differential equation $y' = f(x, y)$ is invariant under the point transformation $(\tilde{x}, \tilde{y}, \frac{d\tilde{y}}{d\tilde{x}})$ iff $\chi^{(1)}(y' - f) = 0$ whenever $y' - f = 0$.

Example: The differential equation, $y' = xy^2$ is invariant under the transformation

$$(\tilde{x}, \tilde{y}) = \left(x, \frac{y}{1 - \epsilon y}\right)$$

To prove this, first compute $\chi^{(1)}$

$$\xi = 0, \quad \eta = y^2 \quad \Rightarrow \quad \chi = y^2 \partial_y$$
$$\eta^{(1)} = \eta_y y' = 2yy' \quad \Rightarrow \quad \chi^{(1)} = y^2 \partial_y + 2yy' \partial_{y'}$$

then apply the theorem.

$$\chi^{(1)}(y' - xy^2) = 2yy' - 2xy^3 = 2y(y' - xy^2)$$

Apply the generator $\chi^{(1)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'}$ to the ODE $y' - f(x, y) = 0$, and replace y' with f wherever it appears.

$$\begin{aligned} 0 = \chi^{(1)}(y' - f) &= \eta^{(1)} - \xi f_x - \eta f_y \\ &= \eta_x + \eta_y y' - y'(\xi_x + \xi_y y') - \xi f_x - \eta f_y \\ &= \eta_x + \eta_y f - f(\xi_x + \xi_y f) - \xi f_x - \eta f_y \end{aligned}$$

Solving PDE is hard, but they are linear $\rightarrow$ method of characteristics.

Finding all possible symmetries for a first order ODE is hard.

Finding symmetries generally becomes easier as the order of the ODE increases.

Software implementations of Lie's method allow you to (mostly) avoid this.

Higher Order Prolongations

Consider a higher order ODE.

$$\frac{d^{n+1}y}{dx^{n+1}} = f\left(x, y, y', y'', \dots, \frac{d^n y}{dx^n}\right)$$

Compute higher order prolongations and their generators.

$$\frac{d^2\tilde{y}}{d\tilde{x}^2} = \frac{d\left(\frac{d\tilde{y}}{d\tilde{x}}\right)}{d\tilde{x}} = \frac{d(y' + \epsilon\eta^{(1)} + o(\epsilon))}{d(x + \epsilon\xi + o(\epsilon))} = y'' + \epsilon\left(\frac{d\eta^{(1)}}{dx} - y''\frac{d\xi}{dx}\right) + o(\epsilon)$$

Therefor

$$\eta^{(2)} = \frac{d\eta^{(1)}}{dx} - y''\frac{d\xi}{dx} = \eta_x^{(1)} + \eta_y^{(1)}y' + \eta_{y'}^{(1)}y'' - y''(\xi_x + \xi_y y')$$

These computations have been automated by computer algebra systems.

Texts list prolongations up to 2nd order, which are relevant for applications.

Lie's theorem applies to $y'' = f(x, y, y')$ if we use the second prolongation.

$$0 = \chi^{(2)}(y'' - f) = \eta^{(2)} - \xi f_x - \eta f_y - \eta^{(1)} f_{y'}$$

Compute the right side via formulas on previous slides, then replace y'' with f.

Example: The equation of motion for a non-accelerating object in 1D, $\ddot{y} = 0$.

$$\eta_{tt} + (2\eta_{ty} - \xi_{tt})\dot{y} + (\eta_{yy} - 2\xi_{ty})\dot{y}^2 - \xi_{yy}\dot{y}^3 = 0$$

This is not as hard as it looks to solve.

Computing the Lie Point Symmetries of $\ddot{y} = 0$

$$\eta_{tt} + (2\eta_{ty} - \xi_{tt})\dot{y} + (\eta_{yy} - 2\xi_{ty})\dot{y}^2 - (\xi_{yy})\dot{y}^3 = 0$$

Recall that $(t, y, \dot{y})$ are viewed as three independent variables. But ξ and η only depend on (t, y).

$$\eta_{tt} = 0, \qquad 2\eta_{ty} - \xi_{tt} = 0, \qquad \eta_{yy} - 2\xi_{ty} = 0, \qquad \xi_{yy} = 0$$

Easily solved, process can be automated.

$$\eta = k_1 + k_2 y + k_3 t + k_4(y^2 + ty)$$
$$\xi = k_5 + k_6 t + k_7 y + k_8(t^2 + ty)$$

Geometers: $\ddot{y} = 0$ is the equation of a line. We've found the point transformations which map lines to lines.

Interpreting the Lie Point Symmetries of $\ddot{y} = 0$

$$\eta = k_1 + k_2 y + k_3 t + k_4 t y + k_8 y^2$$
$$\xi = k_5 + k_6 t + k_7 y + k_8 t y + k_4 t^2$$

There is a set (a group actually) of eight symmetries.

$\chi_1 = \partial_y,$	$(\tilde{t}, \tilde{y}) = (t, y + \epsilon),$	Spatial Translation
$\chi_2 = y\partial_y,$	$(\tilde{t}, \tilde{y}) = (t, e^{\epsilon} y),$	Spatial Scaling
$\chi_3 = t\partial_y,$	$(\tilde{t}, \tilde{y}) = (t, y + \epsilon t),$	Galilean Boost
$\chi_4 = t^2\partial_t + ty\partial_y,$	$(\tilde{t}, \tilde{y}) = \left(\frac{t}{1-\epsilon t}, \frac{y}{1-\epsilon t}\right),$	Projection
$\chi_5 = \partial_t,$	$(\tilde{t}, \tilde{y}) = (t + \epsilon, y),$	Time Translation
$\chi_6 = t\partial_t,$	$(\tilde{t}, \tilde{y}) = (e^{\epsilon} t, y),$	Time Scaling
$\chi_7 = y\partial_t,$	$(\tilde{t}, \tilde{y}) = (t + \epsilon y, y),$	Inversion w/ Galilean Boost
$\chi_8 = ty\partial_t + y^2\partial_y,$	$(\tilde{t}, \tilde{y}) = \left(\frac{t}{1-\epsilon y}, \frac{y}{1-\epsilon y}\right),$	Inversion w/ Projection

The Algebra of Lie Symmetries for $\ddot{y} = 0$

(If you haven't studied algebra, cover your eyes and ears.)

The Lie bracket

$$[\chi_i, \chi_j] = \chi_i \chi_j - \chi_j \chi_i$$

is the infinitesimal generator of the transformation group commutator

$$[g_i, g_j] = g_i^{-1} g_j^{-1} g_i \; g_j \; .$$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
χ_1		χ_1		χ_3			χ_5	$2\chi_2 + \chi_6$
χ_2	$-\chi_1$		$-\chi_3$				χ_7	χ_8
χ_3		χ_3			$-\chi_1$	$-\chi_3$	$\chi_6 - \chi_2$	χ_4
χ_4	$-\chi_3$				$-\chi_2 - 2\chi_6$	$-\chi_4$	$-\chi_8$	
χ_5			χ_1	$\chi_2 + 2\chi_6$		χ_5		χ_7
χ_6			χ_3	χ_4	$-\chi_5$		$-\chi_7$	
χ_7	$-\chi_5$	$-\chi_7$	$\chi_2 - \chi_6$	χ_8		χ_7		
χ_8	$-2\chi_2 - \chi_6$	$-\chi_8$	$-\chi_4$		$-\chi_7$			

Interpreting the Lie Point Symmetries of $\ddot{y} = 0$

In a series of unfriendly letters, Leibniz and one of Newton's supporters, Clarke, argued about physics. When discussing whether space would exist in the absence of mass, they described these symmetries in words:

$\chi_1 = \partial_y,$	$(\tilde{t}, \tilde{y}) = (t, y + \epsilon),$	Spatial Translation
$\chi_2 = y\partial_y,$	$(\tilde{t}, \tilde{y}) = (t, e^{\epsilon} y),$	Spatial Scaling
$\chi_3 = t\partial_y,$	$(\tilde{t}, \tilde{y}) = (t, y + \epsilon t),$	Galilean Boost
$\chi_5 = \partial_t,$	$(\tilde{t}, \tilde{y}) = (t + \epsilon, y),$	Time Translation
$\chi_6 = t\partial_t,$	$(\tilde{t}, \tilde{y}) = (e^{\epsilon} t, y),$	Time Scaling

The model for a non-accelerating object, $\ddot{y} = 0$, is unchanged if:

- The modeler moves to a new fixed location (χ_1).
- The modeler changes the ruler they use to measure distances (χ_2).
- The modeler drifts at constant speed (χ_3).
- The modeler starts their clock at a different time (χ_5).
- The modeler changes the clock they use to measure time intervals (χ_6).

Interpreting the Lie Point Symmetries of $\ddot{y} = 0$

I asked my philosophy professor why Leibniz and Clarke didn't mention these:

$$\chi_4 = t^2\partial_t + ty\partial_y, \quad (\tilde{t}, \tilde{y}) = \left(\frac{t}{1-\epsilon t}, \frac{y}{1-\epsilon t}\right), \quad \text{Projection}$$

$$\chi_7 = y\partial_t, \quad (\tilde{t}, \tilde{y}) = (t + \epsilon y, y), \quad \text{Inversion w/ Galilean Boost}$$

$$\chi_8 = ty\partial_t + y^2\partial_y, \quad (\tilde{t}, \tilde{y}) = \left(\frac{t}{1-\epsilon y}, \frac{y}{1-\epsilon y}\right), \quad \text{Inversion w/ Projection}$$

He didn't have an explanation, but he thought pointing this out to them would have been a great way to poke the bee hive.

(To be fair, the Leibniz/Clark brawl predates Lie by 200 years.)

What Can We Do with Lie Point Symmetries?

- Symmetry can provide insight into a model
  e.g., the projection Clarke and Leibniz may not have known.

- Symmetries can show which quantities are preserved by the model.
  e.g., energy, momentum, area, *etc*.

- Symmetries can be used to create a new coordinate system where...

  The DE is of a simpler type.

  The DE is of lower order.

  The number of independent variables is smaller (PDE).

  The number of dependent variables is smaller (systems).

- Under certain conditions, knowledge of n symmetries of an nth order DE leads to a complete solution using nothing more than a sequence of integrations and coordinate changes.

Theorem (Lie, 1888)

If the ODE

$$y^{(n)} = f\left(x, y, y', \ldots, y^{(n-1)}\right)$$

admits a point symmetry with generator

$$\chi = \xi \partial_x + \eta \partial_y,$$

then there exists **canonical coordinates** $u = u(x, y)$ *and* $v = v(x, y)$*, satisfying*

$$\chi u = 0, \qquad \chi v = 1$$

such that, in this new coordinate system, the ODE is of the form

$$v^{(n)} = g(u, v', \ldots, v^{(n-1)})$$

and the symmetry is simply translation,

$$\chi = \partial_v$$

Q: Why might this new ODE be advantageous?

Canonical Coordinates: Separable ODE

The separable first order ODE

$$\frac{dy}{dx} = f(x)g(y)$$

admits (at least) two symmetries.

$$\chi_1 = \frac{1}{f(x)}\partial_x, \qquad \chi_2 = g(y)\partial_y$$

Each of these suggest the use of new coordinates.

$$\chi_1 \implies (u, v) = \left(y, \int f(x)dx\right) \implies \frac{dv}{du} = \frac{1}{g(u)}$$

$$\chi_2 \implies (u, v) = \left(x, \int \frac{1}{g(y)}dy\right) \implies \frac{dv}{du} = f(u)$$

Either of these may be solved by integrating once.

(You've seen this before, in disguise.)

Canonical Coordinates: Linear ODE

The inhomogeneous first order linear ODE

$$y' + f(x)y = g(x)$$

admits a symmetry of the form

$$\chi = \Phi(x)\partial_y$$

where Φ is any solution of the homogeneous ODE, $\Phi' + f(x)\Phi = 0$.

$$\chi = \Phi(x)\partial_y \quad \Longrightarrow \quad (u, v) = \left(x, \frac{y}{\Phi(x)}\right) \quad \Longrightarrow \quad \frac{dv}{du} = \frac{g(u)}{\Phi(u)}$$

This may be solved by integrating once.

(You've seen this before, in disguise, in more than one form.)

More Insights into Linear ODE

Every homogeneous linear ODE

$$a_n(x)y^{(n)} + \ldots + a_1(x)y' + a_0(x)y = 0$$

admits a symmetry of the form

$$\chi = \phi(x)\partial_y$$

where $\phi(x)$ is any solution. The generator χ corresponds to the transformation

$$\begin{aligned} \tilde{x} &= x \\ \tilde{y} &= y + \epsilon\phi(x) \end{aligned}$$

(You've seen this before, in disguise.)

Canonical Coordinates: Linear ODE

Every homogeneous linear ODE

$$a_n(x)y^{(n)} + \ldots + a_1(x)y' + a_0(x)y = 0$$

admits a symmetry of the form

$$\chi = \phi(x)\partial_y$$

where $\phi(x)$ is any solution.

$$\chi = \phi(x)\partial_y \quad \Longrightarrow \quad (u, v) = (x, y/\phi) \quad \Longrightarrow \quad b_n(u)v^{(n)} + \ldots + b_1(u)v' = 0$$

with $b_k = \sum_{j=k}^{n} \binom{j}{k} a_j \phi^{(j-k)}$.

(You've seen this before, in disguise.)

Canonical Coordinates: Autonomous ODE

Every autonomous ODE

$$y^{(n)} = f\left(y, y', \ldots, y^{(n-1)}\right)$$

is invariant under translations of the independent variable, $\chi = \partial_x$. This suggests a new coordinate system

$$(u, v) = (y, x)$$

in which the ODE is lower order (tough to write general form).

Example: The equation for the displacement of a mass attached to a Hookean spring with quadratic drag

$$my'' = -ky - qy'|y'|$$

transforms to

$$mv'' = ku(v')^3 + q|v'|$$

which is first order in v'.

Canonical Coordinates: Autonomous ODE

The coordinate change

$$(u, v) = (y, x)$$

causes a potential equation

$$my'' + f(y) = 0$$

to be transformed to

$$-m(v')^{-3}v'' + f(u) = 0$$

which integrates to

$$\frac{1}{2}m(v')^{-2} + \int f(u)du = c$$

which physicists will recognize as

$$\text{Kinetic Energy} + \text{Potential Energy} = \text{constant}$$

The heat equation $u_t = u_{xx}$ has the scaling symmetry

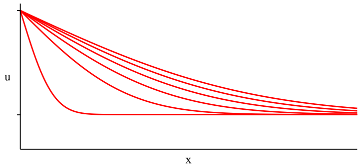
$$(\tilde{t}, \tilde{x}, \tilde{u}) = \left(e^{2\epsilon} t, e^{\epsilon} x, u\right), \qquad \chi = 2t\partial_t + x\partial_x$$

suggesting the new variables $y = \frac{x}{\sqrt{t}}$, $v(y) = u(x, t)$.

$$v'' + \frac{1}{2} y v' = 0$$

Not only is this an ODE (reduced independent variables) but it is first order in v' (reduced order) and is separable (has additional symmetries).

$$u = u_0 + \frac{2(u_1 - u_0)}{\sqrt{\pi}} \int_0^{x/2\sqrt{t}} e^{-z^2} dz$$



Hold one end of a wire at u_0. Heat front propagation at speed $\propto t^{-1/2}$.

The wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

has translational symmetries in each independent variable,

$$\chi = a\partial_x + b\partial_t$$

for any constants (a, b).

This suggests a new variable, $p = bx - at$. Suppose we choose a 2nd such variable, $q = ex - dt$. Are there any especially good choices of (a, b, d, e)?

$$u_{tt} - c^2 u_{xx} = (a^2 - b^2 c^2) u_{pp} + 2(ad - bec^2) u_{pq} + (d^2 - e^2 c^2) u_{qq}$$

Yes: $b = e = 1$, $a = -d = c$.

$$u_{pq} = 0 \qquad \Longrightarrow \qquad u = f(x - ct) + g(x + ct)$$

Fitting initial data gives the famous d'Alembert formula.

The wave equation $u_{tt} - c^2 u_{xx} = 0$ admits the symmetry $\chi = x\partial_t + c^2 t\partial_x$ corresponding to

$$\tilde{x} = x\cosh(\epsilon c) + ct\sinh(\epsilon c)$$
$$\tilde{t} = t\cosh(\epsilon c) + \frac{x}{c}\sinh(\epsilon c)$$

Physicists will recognize this in an alternate form (Lorentz Transformation).

$$\tilde{x} = \frac{x - vt}{\sqrt{1 - (v/c)^2}}$$
$$\tilde{t} = \frac{t - (v/c^2)x}{\sqrt{1 - (v/c)^2}}$$

The symmetry χ leaves $\sqrt{c^2t^2 - x^2}$ invariant (Minkowski Metric).

- Analytical DE solution methods have existed since Newton. The number and variety of methods created in the subsequent 300 years is staggering.
- Lie's genius was recognizing that many analytic methods were special cases of the general theory of invariance and symmetry.
- Some numerical methods with special properties, *e.g.*, geometric integrators, have existed since at least the 1980s.
- Work of the past decade suggests that these methods are special cases of a general theory of invariant numerical schemes.

The heat equation, $u_t = u_{xx}$, admits eight distinct symmetries.

χ	$(\tilde{t}, \tilde{x}, \tilde{u})$	Meaning
∂_t	$(t+\epsilon, x, u)$	Time Translation
∂_x	$(t, x+\epsilon, u)$	Space Translation
∂_u	$(t, x, u+\epsilon)$	Temp. Translation
$u\partial_u$	$(t, x, e^{\epsilon}u)$	Temp. Scaling
$2t\partial_t + x\partial_x$	$(e^{2\epsilon}t, e^{\epsilon}x, u)$	Similarity Scaling
$\phi(x,t)\partial_u$	$(t, x, u+\epsilon\phi)$	Superposition
$t\partial_x - \frac{xu}{2}\partial_u$	$\left(t, x+\epsilon t, ue^{\frac{x^2-(x+\epsilon t)^2}{4t}}\right)$	Galilean Boost
$t^2\partial_t + xt\partial_x - \frac{x^2u+2ut}{4}\partial_u,$	$\left(\frac{t}{1-\epsilon t}, \frac{x}{1-\epsilon t}\frac{u\sqrt{1-\epsilon t}}{e^{\frac{x^2\epsilon}{4(1-\epsilon t)}}}\right)$	Projection

Question for algebraists: Does a given Lie algebra uniquely determine a PDE?

Novel Derivation of the Heat Equation

Begin with a general 2nd order in space, 1st order in time PDE:

$$F_0(t, x, u, u_t, u_x, u_{xx}) = 0$$

Require the Three Translational Symmetries:

$$F_1(u_t, u_x, u_{xx}) = 0$$

Require Scaling Symmetry in u:

$$F_2\left(\frac{u_t}{u_x}, \frac{u_t}{u_{xx}}\right) = 0$$

Require Similarity Scaling Symmetry in (t, x):

$$F_3\left(\frac{u_t}{u_{xx}}\right) = 0$$

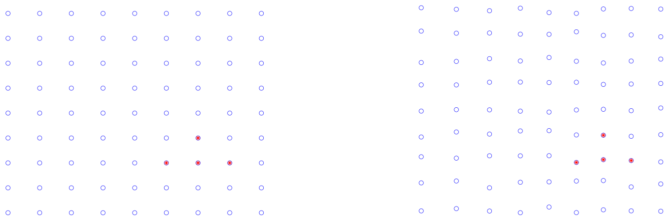
So $u_t = \kappa u_{xx}$, where κ is any root of F_3.

Audience Challenge: Find a weirder derivation of the heat equation.

Finite Difference Methods For the Heat Equation

Create a mesh grid of (x, t) values, with time levels indexed by n and spatial levels indexed by j. Approximate the solution at each grid point.

$$u_j^n \approx u(t_j^n, x_j^n)$$



Choose a stencil. Simplest useful example has 12 elements:

$$\left\{(t_{j-1}^n, x_{j-1}^n, u_{j-1}^n), (t_j^n, x_j^n, u_j^n), (t_{j+1}^n, x_{j+1}^n, u_{j+1}^n), (t_j^{n+1}, x_j^{n+1}, u_j^{n+1})\right\}$$

Find equation involving these variables which approximates the heat equation:

$$F\left(t_p^q, x_p^q, u_p^q\right) = 0$$

Constructing an Invariant Scheme for the Heat Equation

Begin with an equation in 12 functionally independent variables:

$$F_0\left(t_p^q, x_p^q, u_p^q\right) = 0$$

Require the Three Translational Symmetries:

$$F_1\left(\ t_p^q - t_a^b,\ x_p^q - x_a^b,\ u_p^q - u_a^b\ \right) = 0$$

Require Scaling Symmetry in u:

$$F_2\left(\ t_p^q - t_a^b,\ x_p^q - x_a^b,\ \frac{u_p^q - u_a^b}{u_{p'}^{q'} - u_{a'}^{b'}}\right) = 0$$

Require Similarity Scaling Symmetry in (t, x):

$$F_3\left(\frac{(x_p^q - x_a^b)^2}{t_{p'}^{q'} - t_{a'}^{b'}},\ \frac{x_p^q - x_a^b}{x_{p'}^{q'} - x_{a'}^{b'}},\ \frac{u_p^q - u_a^b}{u_{p'}^{q'} - u_{a'}^{b'}}\right) = 0$$

Despite appearances, there are only 7 functionally independent variables.

Constructing an Invariant Scheme for the Heat Equation

All methods invariant under translations and scalings using this stencil depend on 7 functionally independent variables.

$$F_3\left(\frac{(x_p^q - x_a^b)^2}{t_{p'}^{q'} - t_{a'}^{b'}},\ \frac{x_p^q - x_a^b}{x_{p'}^{q'} - x_{a'}^{b'}},\ \frac{u_p^q - u_a^b}{u_{p'}^{q'} - u_{a'}^{b'}}\right) = 0$$

Special Case: Static Uniform Grid.

- All space steps equal and independent of time, $x_j^n = x_j = x_0 + jh$.
- All time steps equal and independent of space, $t_j^n = t^n = t^0 + nk$.

There are now only three functionally independent variables.

$$F_4\left(\frac{h^2}{k},\ \frac{u_{j+1}^n - u_j^n}{u_j^{n+1} - u_j^n},\ \frac{u_j^n - u_{j-1}^n}{u_j^{n+1} - u_j^n}\right) = 0$$

Constructing an Invariant Scheme for the Heat Equation

Which functions of these three invariants will approximate the heat equation?

$$F(I_1, I_2, I_3) = F\left(\frac{h^2}{k}, \ \frac{u_{j+1}^n - u_j^n}{u_j^{n+1} - u_j^n}, \ \frac{u_j^n - u_{j-1}^n}{u_j^{n+1} - u_j^n} \right) = 0$$

Taylor expansion:

$$u_j^{n+1} \approx u(t^{n+1}, x_j) = u(t^n + k, x_j) = u(t^n, x_j) + k u_t(t^n, x_j) + \frac{1}{2} k^2 u_{tt}(t^n, x_j) + \dots$$

$$u_{j-1}^n \approx u(t^n, x_{j-1}) = u(t^n, x_j - h) = u(t^n, x_j) - h u_x(t^n, x_j) + \frac{1}{2} k^2 u_{xx}(t^n, x_j) + \dots$$

$$u_{j+1}^n \approx u(t^n, x_{j+1}) = u(t^n, x_j + h) = u(t^n, x_j) + h u_x(t^n, x_j) + \frac{1}{2} k^2 u_{xx}(t^n, x_j) + \dots$$

Plug into the invariants:

$$I_2 = \frac{u_{j+1}^n - u_j^n}{u_j^{n+1} - u_j^n} \approx \frac{h u_x + \frac{1}{2} h^2 u_{xx} + \dots}{k u_t + \frac{1}{2} k^2 u_{tt} + \dots}, \quad I_3 = \frac{u_j^n - u_{j-1}^n}{u_j^{n+1} - u_j^n} \approx \frac{h u_x - \frac{1}{2} h^2 u_{xx} + \dots}{k u_t + \frac{1}{2} k^2 u_{tt} + \dots}$$

To approximate $u_t = u_{xx}$, the u_x term should be eliminated.

$$I_2 - I_3 \approx \frac{h^2 u_{xx} + \dots}{k u_t + \frac{1}{2} k^2 u_{tt} + \dots} = \frac{h^2 u_{xx}}{k u_t} + \dots = I_1 \frac{u_{xx}}{u_t} + \dots = I_1 + \dots$$

The invariantized scheme is $I_2 - I_3 - I_1 = 0$.

$$\frac{u_{j+1}^n - u_j^n}{u_j^{n+1} - u_j^n} - \frac{u_j^n - u_{j-1}^n}{u_j^{n+1} - u_j^n} - \frac{h^2}{k} = 0$$

or in a more familiar form:

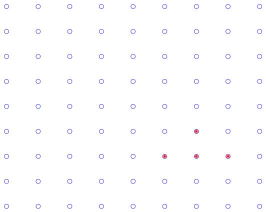
$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} \qquad \longleftrightarrow \qquad u_t = u_{xx}$$

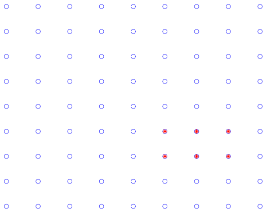
This is the first method in any book on differencing the heat equation. For implementation, it's usually written.

$$u_j^{n+1} = r u_{j-1}^n + (1 - 2r) u_j^n + r u_{j+1}^n$$

with $r = k/h^2 = 1/I_1$. Stable only if $r < 1/2$; usually held constant as $k, h \to 0$.

Repeat the invariance procedure on a larger stencil.





Crank-Nicolson

$$-r u_{j-1}^{n+1} + (2+2r)u_j^{n+1} - r u_{j+1}^{n+1} = r u_{j-1}^{n} + (2-2r)u_j^{n} + r u_{j+1}^{n}$$

Douglas

$$\begin{aligned}(1/6-r)u_{j-1}^{n+1} + (5/3+2r)u_j^{n+1} + (1/6-r)u_{j+1}^{n+1} \\ = (1/6+r)u_{j-1}^{n} + (5/3-2r)u_j^{n} + (1/6+r)u_{j+1}^{n}\end{aligned}$$

Both unconditionally stable, Douglas more accurate (often overlooked).

Crank-Nicolson, $O(k^2 + h^2)$.

$$-ru_{j-1}^{n+1} + (2+2r)u_j^{n+1} - ru_{j+1}^{n+1} = ru_{j-1}^n + (2-2r)u_j^n + ru_{j+1}^n$$

Douglas, $O(k^2 + h^4)$.

$$\begin{aligned}(1/6 - r)u_{j-1}^{n+1} + (5/3 + 2r)u_j^{n+1} + (1/6 - r)u_{j+1}^{n+1} \\ = (1/6 + r)u_{j-1}^n + (5/3 - 2r)u_j^n + (1/6 + r)u_{j+1}^n\end{aligned}$$

These are not the only invariant methods possible using this stencil and mesh.

Theorem

Among all possible linear finite difference schemes on this stencil which are invariant under translations and scalings on a static uniform mesh, Douglas' scheme is optimal with respect to the order of accuracy.

Invariantizing Schemes

A general procedure for producing *invariantized* finite difference methods.

Symmetry-Preserving Numerical Schemes, Bihlo and Valiquette, 2017:

- Compute symmetries of a DE.
- Select a subset of the symmetries for the numerical scheme to also exhibit.
- Use symmetries to produce sequentially smaller sets of invariants.
- Utilize the final set of invariants to approximate the DE.

Some methods produced this way are classical, but others are new and useful, including those with adaptive meshes.

Numerical experiments suggest some of these new schemes are more accurate than classical ones. This field is awaiting rigorous numerical analysis.

Finite element methods in development (Bihlo, Jackaman, & Valiquette, 2020).

Questions for Future Investigators

- What other DE methods do you know? Are they the result of symmetry?
- Will numerical analysts and algebraists team up to prove some theorems?
- What numerical methods will be inspired by Lie-Bäcklund Symmetries, Contact Symmetries, Potential Symmetries, and Discrete Symmetries?
- Finite difference methods give the approximate solution of the original PDE but give the exact solution of the *modified PDE*. Can the symmetries of the modified PDE be computed and what will they tell us?
- Some symmetry preserving methods for ODE give the exact solution (no numerical truncation error!) can we do this for some PDE also?

What Questions Do You Have?

Dedication

This presentation is dedicated to the memory of Professor Jerrold E. Marsden, one of the kindest and most patient men I've ever known. He provided guidance and freedom to an undergraduate interested in symmetry.

The exact ODE

$$\phi_x(x,y) + \phi_y(x,y)\frac{dy}{dx} = f(x)$$

has symmtry

$$\chi = \frac{1}{\phi_y(x,y)}\partial_y$$

suggesting the new coordinates

$$(u,v) = (x, \phi(x,y))$$

in which the ODE becomes

$$\frac{dv}{du} = f(u)$$

which is solved by integrating once.

Why is $\chi = \xi\partial_x + \eta\partial_y$ called a generator?

Theorem

$$\tilde{x} = e^{\epsilon\chi} x$$

Remark: Recall the matrix exponential evolves the initial data along the solutions of $x' = Ax$.

$$x(t) = e^{tA} x(0)$$

χ generates the symmetry $\tilde{x}(x, \epsilon)$ in a similar way.

$$\tilde{x}(x, \epsilon) = e^{\epsilon\chi} \tilde{x}(x, 0)$$

Is separability of a non-linear PDE an indication of underlying symmetry?

Consider 1st order.

$$F_1(t, x, u, u_t, u_x) = 0$$

Insert $u(x, t) = X(x)T(t)$, solve for $X'(x)$ and separate.

$$\begin{aligned} X' &= F_2(t, x, T, X, \dot{T}) \\ &= F_3(x, X) * F_4(t, T, \dot{T}) \end{aligned}$$

Does $X' = F_3(x, X)F_4(t, T, \dot{T})$ exhibit a symmetry for all smooth F_3, F_4?

Conjecture: Not a point symmetry,

$$\chi = \tau(t, x, X, T)\partial_t + \xi(t, x, X, T)\partial_x + \eta_x(t, x, X, T)\partial_X + \eta_t(t, x, X, T)\partial_T$$

but a contact symmetry.

$$\begin{aligned} \chi = &\tau(t, x, X, T, X', T')\partial_t + \xi(t, x, X, T, X', T')\partial_x \\ &+ \eta_x(t, x, X, T, X', T')\partial_X + \eta_t(t, x, X, T, X', T')\partial_T \end{aligned}$$

To solve a Cauchy problem for a quasi-linear first order PDE

$$a(t,x,u)u_t + b(t,x,u)u_x = c(t,x,u)$$

introduce an auxiliary variable s and solve the system

$$\frac{dt}{ds} = a(t,x,u), \qquad \frac{dx}{ds} = b(t,x,u), \qquad \frac{du}{ds} = c(t,x,u)$$

Is the effectiveness of this new variable due to an underlying symmetry?

Conjecture: It's not due to a point symmetry,

$$\chi = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$$

but instead a contact symmetry.

$$\chi = \tau(t,x,u,u_x,u_t)\partial_t + \xi(t,x,u,u_x,u_t)\partial_x + \eta(t,x,u,u_x,u_t)\partial_u$$

Appendix: Other Invariantized Numerical Methods

Burgers, KDV